St George Girls High School

**2021** TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

# Mathematics Extension 2

**General Instructions**

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and/or calculations

**Total marks: 100**

**Section I – 10 marks** (pages 2–5)

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II – 90 marks** (pages 6–13)

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

**Section I**

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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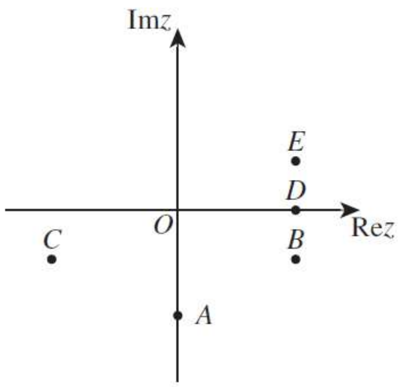
**1** Consider the argand diagram, where $z = a + ib$.




Which of the following pairs of points in the complex plane could represent the square roots of $z$?

A. $A$ and $D$

B. $B$ and $C$

C. $B$ and $E$

D. $C$ and $E$

**2** Which expression is equal to $\int xe^{\frac{x}{2}}\,dx$?

A. $\frac{1}{2}xe^{\frac{x}{2}} - \frac{1}{4}e^{\frac{x}{2}} + C$

B. $\frac{1}{2}xe^{\frac{x}{2}} - \frac{1}{2}e^{\frac{x}{2}} + C$

C. $2xe^{\frac{x}{2}} - 2e^{\frac{x}{2}} + C$

D. $2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + C$

**3** Which of the following is a valid counter-example to the claim:

'If it can purr or roar, it's a cat.'

A. A cat that can purr but not roar.

B. A cat that can't purr or roar.

C. A dog that can roar.

D. A dog that can't purr or roar.

**4** A sphere has equation $(x-k)^2+(y-2k)^2+(z-3k)^2=49$.

For what value of $k$ does the sphere pass through the origin?

A. $7$

B. $\sqrt{7}$

C. $\dfrac{\sqrt{7}}{2}$

D. $\sqrt{\dfrac{7}{2}}$

**5** Given that $a$ and $b$ are positive real numbers, and $a+b=4$, find the minimum value of $\left(1+\dfrac{1}{a}\right)\left(1+\dfrac{1}{b}\right)$.

A. $0$

B. $\dfrac{3}{2}$

C. $4$

D. $\dfrac{9}{4}$

**6** For a certain complex number $z$, it is known that $|z| = 1$.

Which of the following statements must also be true?

A. $z^{-1} = \bar{z}$

B. $Arg(z^{-1}) = Arg(\bar{z})$

C. $|z^{-1}| > |\bar{z}|$

D. $z^{-1} = i\bar{z}$

**7** Which of the following is the Cartesian equation of the line joining $(1, -1, 2)$ and $(-1, 1, 0)$?

A. $x = -y = z - 1$

B. $x - 1 = y + 1 = z - 2$

C. $\dfrac{x+1}{2} = \dfrac{y-1}{2} = \dfrac{z-2}{2}$

D. $\boldsymbol{r} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

**8** Which of the following is a FALSE statement, given $x, y \in \mathbb{Z}$?

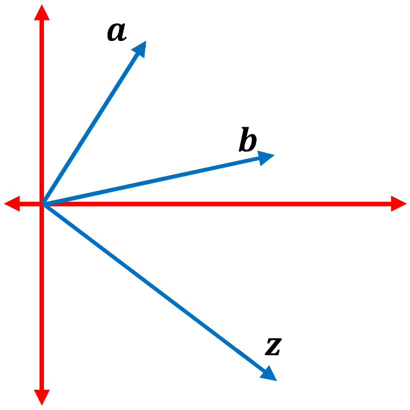
A. $\forall x \exists y\ (x - y = 0)$

B. $\forall x \exists y\ (3x - y = 0)$

C. $\forall x \exists y\ (x - 3y = 0)$

D. $\exists x \exists y\ (x + y = 8)$

**9** The vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{z}$ are shown below.




Which of the following could be equivalent to $\boldsymbol{z}$, where $\lambda_1, \lambda_2 \in \mathbb{R}$?

A. $\lambda_1 \boldsymbol{a} + \lambda_2 \boldsymbol{b}$

B. $\boldsymbol{a} + \lambda_2 \boldsymbol{b}$

C. $\lambda_1 \boldsymbol{a} + \boldsymbol{b}$

D. None of the above

**10** How many distinct roots has the equation $(z^4 - 1)(z^2 - 3iz - 2) = 0$, where $z \in \mathbb{C}$?

A. 3

B. 4

C. 5

D. 6

## Section II

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours and 45 minutes for this section**

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks)

(a) Consider the vectors $\boldsymbol{u} = 4\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - 2\boldsymbol{k}$.

(i) Find the vector $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$. **1**

(ii) Find the exact value of $|\boldsymbol{u}|$. **1**

(iii) Hence find the vector $\boldsymbol{t}$ with magnitude 4, in the direction opposite to $\boldsymbol{u}$. **1**

(b) (i) Find values of $a$ and $b$ such that **2**

$$\frac{9x-2}{(2x+1)(x-6)} = \frac{a}{(2x+1)} + \frac{b}{(x-6)}.$$

(ii) Hence or otherwise find **2**

$$\int \frac{9x-2}{(2x+1)(x-6)}\,dx.$$

(c) Consider the complex number $z = 1 - \sqrt{3}i$

(i) Evaluate $z^{-1}$. **1**

(ii) Express $z$ in the form $e^{i\theta}$. **2**

(iii) Hence, or otherwise, evaluate $z^9$. **1**

(iv) Find $\ln z$, in the form $a + ib$, where $a, b \in \mathbb{R}$. **2**

**Question 11** (continued)

(d) Consider the proposition:

*I won't buy your fruit if it's not sweet.*

(i) State the contrapositive of the proposition. **1**

(ii) State a possible counterexample to the proposition. **1**

**End of Question 11**

**Question 12** (15 marks)

(a) (i) Find the square root(s) of $7 - 24i$, in the form $x + iy$, where $x, y \in \mathbb{R}$. **3**

(ii) Hence or otherwise solve $2z^2 + 6z + (1 + 12i) = 0$, for $z \in \mathbb{C}$. **2**

(b) Find the angle between $\boldsymbol{u} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{v} = \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$, correct to the nearest degree. **2**

(c) (i) Show that $\omega = 2i$ is a solution to $\omega^6 = -64$. **1**

(ii) Hence find the other distinct solutions to $\omega^6 = -64$, where $\omega \in \mathbb{C}$. **2**

(d) Use proof by contraposition to prove that if $3k + 1$ is even, then $k$ is odd, $\forall k \in \mathbb{Z}$ **2**

(e) Find the exact value of $\displaystyle\int_0^1 \sin^{-1} x \, dx$. **3**

**End of Question 12**

**Question 13** (15 marks)

(a) Use proof by contradiction to prove that $\log_6 12$ is irrational. **3**

(b) Find $\displaystyle\int \frac{x^2}{x^2+2x+10}\,dx$. **3**

(c) A subset of the complex plane is defined by the relation $|z-(4-3i)| \leq 3$.

(i) Draw a sketch of this relation. **1**

(ii) Given that $z$ is a complex number that satisfies the relation, find the minimum and maximum values of $|z|$. **2**

(iii) Given that $z$ is a complex number that satisfies the relation, find the minimum and maximum values of $\operatorname{Arg}(z)$. **2**

(d) A curve is defined parametrically as $\boldsymbol{r} = (\cos t)\boldsymbol{i} + \boldsymbol{j} + (\sin^2 t)\boldsymbol{k}$, for parameter $t$, $0 \leq t \leq 2\pi$.

(i) Explain why this curve is two-dimensional. **1**

(ii) Sketch the curve, clearly labelling all important features, including the coordinate axes. **3**

**End of Question 13**

**Question 14** (15 marks)

(a) It can be shown that $\dfrac{a+b}{2} \geq \sqrt{ab}$. DO NOT PROVE THIS.

(i) Prove that $(a+b)(b+c)(c+a) \geq 8abc$ for all positive real numbers $a$, $b$, and $c$. **1**

(ii) Suppose that $x$, $y$, and $z$ are the lengths of the sides of a triangle. Using the result from part (i), deduce that **2**

$$xyz \geq (y+z-x)(z+x-y)(x+y-z)$$

(b) (i) Find the constants $A$ and $B$ such that **2**

$$\frac{1}{\cos x} = \frac{A\cos x}{1-\sin x} + \frac{B\cos x}{1+\sin x}$$

(ii) Hence find the exact value of $\displaystyle\int_0^{\frac{\pi}{6}} \sec x\, dx$. **3**

(c) Prove that the following three vectors CANNOT be the three sides of a triangle: **2**
$\boldsymbol{t} = 4\boldsymbol{i} - \boldsymbol{k} - 2\boldsymbol{j}$, $\boldsymbol{u} = 5\boldsymbol{i} - 2\boldsymbol{j} + 7\boldsymbol{k}$, and $\boldsymbol{v} = \boldsymbol{i} + 3\boldsymbol{k} - 2\boldsymbol{j}$

(d) It can be shown that $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$. **3**

DO NOT PROVE THIS.

Use this property to find the exact value of $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}$.

**Question 14** (continued)

(e) Jimin claims that for the function $f(n) = n^2 + n + 41$, $f(n)$is prime for all positive integers $n$.

For example, $f(1) = 43, f(2) = 47, f(3) = 53$ and so on.
DO NOT PROVE THIS.

(i) Jungkook attempts to disprove this claim by providing the following counter-example: **1**

"7 is a prime number, but 7 isn't generated by your formula."

Explain why this is not a valid counter-example.

(ii) State a counter-example that disproves Jimin's claim. **1**

**End of Question 14**

**Question 15** (15 marks)

(a) Let $I_n = \int x(\ln x)^n \, dx$.

(i) Show that $I_n = \frac{1}{2}x^2(\ln x)^n - \frac{1}{2}nI_{n-1}$. **2**

(ii) Hence evaluate $\int_1^3 x(\ln x)^2 \, dx$. **3**

(b) The lines $l_1$ and $l_2$ have vector equations as follows:

$$\boldsymbol{l_1} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \quad \boldsymbol{l_2} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$$

(i) Show that $l_1$ and $l_2$ DO NOT intersect. **3**

(ii) It can be shown that **4**
$a^2 + b^2 + c^2 \equiv (a + b + c)^2 - 2(ab + ac + bc)$
for $a, b, c \in \mathbb{R}$. DO NOT PROVE THIS.

Using this identity, or otherwise, show that the distance $d$ between a point on $l_1$and a point on $l_2$ is given by

$$d = \sqrt{(3\lambda_2 - 4\lambda_1 - 5)^2 + (\lambda_1 - 1)^2 + 36}$$

(iii) Hence determine the minimum distance between $l_1$ and $l_2$. **1**

(iv) Find the coordinates of the points on the two lines that are the minimum distance apart. **2**

**End of Question 15**

**Question 16** (15 marks)

(a) Prove the following statement for all r, where $r \in \mathbb{R}$: **2**

*If r is irrational, then $\sqrt{r}$ is irrational.*

(b) By evaluation or otherwise, prove that $i^i$ is a real number. **2**

(c) Let $t = \tan\frac{x}{2}$.

(i) Show that $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$. **1**

(ii) Show that $\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \sin x$. **2**

(iii) Hence, show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}\,dx = \frac{2}{\sqrt{1-k^2}}\tan^{-1}\sqrt{\frac{1-k}{1+k}}$ **4**

where $0 < k < 1$.

(iv) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{2+\sin x}\,dx$, where $n = 0, 1, 2, \ldots.$ **1**

Show that $I_{n+1} + 2I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx$.

(v) Hence, or otherwise, find the value of $I_2$. Give your answer in the form $m\pi + 1$, where $m$ is irrational. **4**

**End of paper**

Multiple Choice

1. Roots evenly spaced $\therefore$ C & E $\therefore$ D

2. $\int x e^{\frac{x}{2}} dx$

$u = x \quad v = 2e^{\frac{x}{2}}$

$u' = 1 \quad v' = e^{\frac{x}{2}}$

$= uv - \int u'v\,dx$

$= 2xe^{\frac{x}{2}} - \int 2e^{\frac{x}{2}} dx$

$= 2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + C \quad \therefore$ D

3. C

4. $(k)^2 + (2k)^2 + (3k)^2 = 49$

$14k^2 = 49$

$k^2 = \frac{49}{14}$

$= \frac{7}{2}$ D

$K = \sqrt{\frac{7}{2}}$

5. Minimum of $(1+\frac{1}{a})(1+\frac{1}{b})$ when neither a nor b are very small

since $a+b=4$, try $a=b=2$ OR

$(1+\frac{1}{2})(1+\frac{1}{2}) = \frac{3}{2} \times \frac{3}{2}$

$= \frac{9}{4} \quad \therefore$ D

6. $z^{-1} = \frac{1}{z}$

$= \frac{\bar{z}}{z\bar{z}}$

$= \frac{\bar{z}}{|z|^2}$

$= \bar{z}$ (since $|z| = 1$) $\therefore$ A

7. $d = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\therefore \frac{x-a}{1} = \frac{y-b}{-1} = \frac{z-c}{1}$

passes through $(0,0,1)$ so

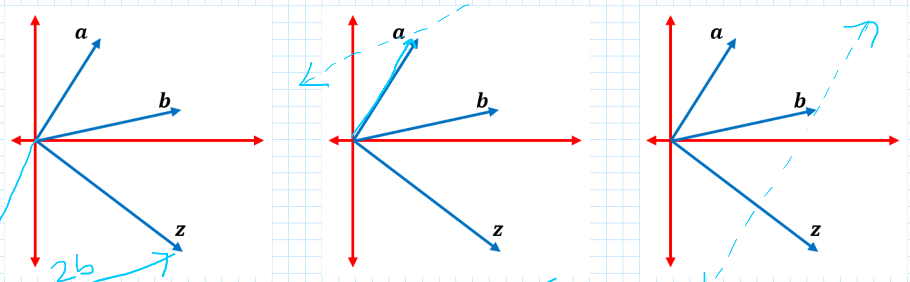
$\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-1}{1}$

$\therefore x = -y = z-1$ A

8. C Eg if $x = 1$, $1 - 3y = 0$ ,

$y = \frac{1}{3} \notin \mathbb{Z}$

9




$-2a$ $\quad$ $2b$

$\lambda_1 a + \lambda_2 b$ ✓ $\qquad a + \lambda_2 b$ ✗ $\qquad \lambda_1 a + b$ ✗

$\therefore$ A

10 $\quad (z^4-1)(z^2-3(z-2))=0$

$(z^2-1)(z^2+1)(z-2)(z-1)=0$

$(z-1)^2(z+1)(z-2)(z^2+1)=0$

$\therefore$ 5 distinct roots

$\therefore$ C

Question 11

a) i $\underset{\sim}{w} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k} + \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$= 5\underset{\sim}{i} + \underset{\sim}{k}$

ii $|u| = \sqrt{4^2 + 2^2 + 3^2}$

$= \sqrt{16 + 4 + 9}$

$= \sqrt{29}$

iii $t = -4 \times \hat{\underset{\sim}{u}}$

$= \frac{-4}{29}(4i - 2j + 3k)$

$$\frac{1}{(2x+1)(\cancel{x-6})} + \frac{4}{\cancel{x+6}\,(x-6)} = \frac{4x - 6 + 8x + 4}{(2x+1)(x-6)}$$

$= 12x - 2$

b) i $\frac{9x-2}{(2x+1)(x-6)} = \frac{a}{2x+1} + \frac{b}{x-6}$

$9x - 2 = a(x-6) + b(2x+1)$

when $x = 6$, $52 = 13b$

$b = 4$

when $b = 4$, $a = 1$

ii $\int \frac{9x-2}{(2x-1)(x-6)}dx = \int \frac{1}{2x+1} + \frac{4}{x-6}\,dx$

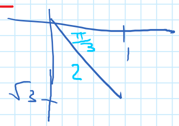
$= \frac{1}{2}\ln|2x+1| + 4\ln|x-6| + C$

c) i $z^{-1} = \frac{1}{1 - \sqrt{3}i}$

$= \frac{1 + \sqrt{3}i}{4}$

$= \frac{1}{4} + \frac{\sqrt{3}}{4}i$

ii




$z = 2e^{i\frac{\pi}{3}}$

iii $z^9 = \left(2e^{i\frac{\pi}{3}}\right)^9$

$= 512\, e^{i\frac{9\pi}{3}}$

$= 512\, e^{i3\pi}$

$= -512$

iv $\ln z = \ln\left(2e^{i\frac{\pi}{3}}\right)$

$= \ln 2 + \ln e^{i\frac{\pi}{3}}$

$= \ln 2 + i\frac{\pi}{3}$

d) i If I buy your fruit, it's sweet.

ii Buying fruit that's not sweet.

Question 12

a) i let $x+iy = \sqrt{7-24i}$

$(x+iy)^2 = 7-24i$

$x^2-y^2+2xyi = 7-24i$

$\therefore x^2-y^2=7$ $\quad$ $2xy=-24$

$xy=-12$

$\therefore x=4, y=-3$ or $x=-4, y=3$

$\therefore 4-3i$ and $-4+3i$

ii $2z^2+6z+(1+12i)=0$

$\Delta = b^2-4ac$

$= 36-4\times 2\times(1+12i)$

$= 36-8-96i$

$= 28-96i$

$= 4(7-24i)$

$\therefore z = \dfrac{-6 \pm \sqrt{4(7-24i)}}{4}$

$= \dfrac{-6 \pm 2(4-3i)}{4}$

$= \dfrac{-3 \pm (4-3i)}{2}$

$= \dfrac{1-3i}{2}, \dfrac{-7+3i}{2}$

b) i $u\cdot v = |u||v|\cos\theta$

$\therefore \cos\theta = \dfrac{u\cdot v}{|u||v|}$

$= \dfrac{1-4-9}{|\sqrt{14}||\sqrt{14}|}$

$= \dfrac{-12}{14}$

$\therefore \theta = 148.997\ldots$

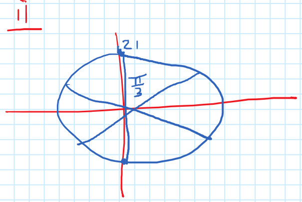
$\doteqdot 149^\circ$

c) i $(2i)^6 = 2^6 i^6$

$= 64\times -1$

$= -64$

ii




$\therefore -2i, 2\operatorname{cis}\frac{\pi}{6}, 2\operatorname{cis}\left(-\frac{\pi}{6}\right), 2\operatorname{cis}\frac{5\pi}{6}, 2\operatorname{cis}\left(-\frac{5\pi}{6}\right)$

d) If $3k+1$ is even, then $k$ is odd

contrapositive: If $k$ is even, then $3k+1$ is odd

let $k=2a$, $a\in\mathbb{Z}$

$\therefore 3k+1 = 3(2a)+1$
$= 6a+1$
$= 2(3a)+1$

which is odd, since $2(3a)$ is even $(a\in\mathbb{Z})$

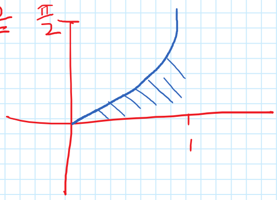
e) $\int_0^1 \sin^{-1}x\,dx = \left[x\sin^{-1}x\right]_0^1 - \int \frac{x}{\sqrt{1-x^2}}\,dx$ $\quad u=\sin^{-1}x \quad v=x$, $u'=\frac{1}{\sqrt{1-x^2}} \quad v'=1$

$= 1\sin^{-1}(1) + \frac{1}{2}\int_0^1 -2x\,(1-x^2)^{-\frac{1}{2}}\,dx$

$= \frac{\pi}{2} - 1\left[2\sqrt{1-x^2}\right]_0^1$

$= \frac{\pi}{2} + [0-1]$

$= \frac{\pi}{2} - 1$

OR 


$A = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x\,dx$

$= \frac{\pi}{2} - 1$

Question 13

a) Assume $\log_6 12$ is rational

$\therefore \log_6 12 = \frac{a}{b}$ where $a, b \in \mathbb{Z}^+$ and $a, b$ have no common factors other than one.

$6^{\frac{a}{b}} = 12$

$6^a = 12^b$

$(2\times 3)^a = (2\times 2\times 3)$

$2^a \times 3^a = 2^{2b} \times 3^b$

$\therefore a = 2b$ and $a = b$

which is a contradiction

$\therefore \log_6 12$ is irrational.

b) $\int \frac{x^2}{x^2+2x+10}\,dx = \int \frac{x^2+2x+10}{x^2+2x+10} - \frac{2x+2}{x^2+2x+10} - \frac{8}{x^2+2x+10}\,dx$

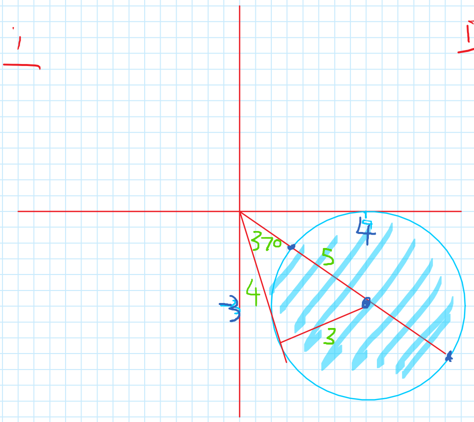
$= x - \ln|x^2+2x+10| - 8\int \frac{1}{x^2+2x+1+9}\,dx$

$= x - \ln|x^2+2x+10| - 8\int \frac{1}{(x+1)^2+3^2}\,dx$

$= x - \ln|x^2+2x+10| - \frac{8}{3}\tan^{-1}\frac{x+1}{3} + C$

c)

i




ii max $|z| = 5+3$

$= 8$

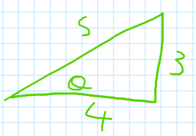
min $|z| = 5-3$

$= 2$

iii max $\arg(z) = 0$

min $\arg(z) \doteq 2\times 37°$

$= 74°$




$\tan\theta = \frac{3}{4}$

$\theta = 37°$

d) i The y-component is always 1, so the curve exists in the plane $y = 1$.

ii $x = \cos t$ $\quad z = \sin^2 t$

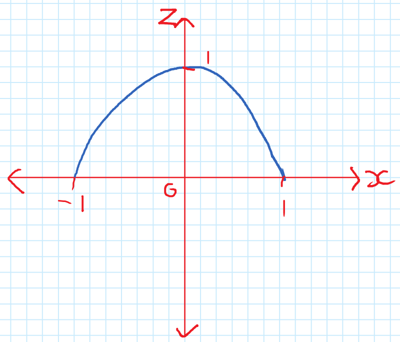
$= 1 - \cos^2 t$

$= 1 - x^2$

since $0 \le t \le 2\pi$

and $x = \cos t$,

$-1 \le x \le 1$



Question 14

a) i

$a+b \geq 2\sqrt{ab}$

$b+c \geq 2\sqrt{bc}$

$c+a \geq 2\sqrt{ac}$

$$\therefore (a+b)(b+c)(a+c) \geq 2\sqrt{ab} \times 2\sqrt{bc} \times 2\sqrt{ac}$$
$$= 8\sqrt{ab \times bc \times ac}$$
$$= 8\sqrt{a^2b^2c^2}$$
$$= 8abc$$

ii

let $a+b=x$, $b+c=y$, $c+a=z$

$\therefore b = x-a$, $c = y-b$, $a = z-c$

$\Rightarrow b = x-(z-c)$, $c = y-(x-a)$, $a = z-(y-b)$

$\Rightarrow b = x-(z-(y-b))$, $c = y-(x-(z-c))$, $a = z-(y-(x-a))$

$\Rightarrow 2b = x-z+y$, $2c = y-x+z$, $2a = z-y+x$

$\Rightarrow b = \dfrac{x+y-z}{2}$, $c = \dfrac{z+y-x}{2}$, $a = \dfrac{x+z-y}{2}$

Since $(a+b)(b+c)(c+a) \geq 8abc$

then $(x)(y)(z) \geq 8\left(\dfrac{x+z-y}{2}\right)\left(\dfrac{x+y-z}{2}\right)\left(\dfrac{z+y-x}{2}\right)$

$\therefore xyz \geq (x+z-y)(x+y-z)(z+y-x)$

b) i

$$\frac{1}{\cos x} = \frac{A\cos x}{1-\sin x} + \frac{B\cos x}{1+\sin x}$$

$$\frac{\cos^2 x}{\cos x} = A\cos x(1+\sin x) + B\cos x(1-\sin x)$$

$$1 = A(1+\sin x) + B(1-\sin x)$$

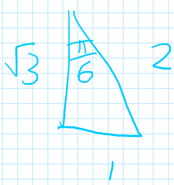
$\therefore A+B=1$

when $x=\frac{\pi}{2}$, $1 = A(1+1)+B(1-1)$

$2A=1$ $\quad \therefore A=B=\frac{1}{2}$

ii

$$\int_0^{\frac{\pi}{6}} \sec x \, dx = \int_0^{\frac{\pi}{6}} \frac{\frac{1}{2}\cos x}{1-\sin x} + \frac{\frac{1}{2}\cos x}{1+\sin x}$$




$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x} - \frac{-\cos x}{1-\sin x}$$

$$= \frac{1}{2}\Big[\ln|1+\sin x| - \ln|1-\sin x|\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{2}\Big[\left(\ln\left|1+\tfrac{1}{2}\right| - \ln\left|1-\tfrac{1}{2}\right|\right) - (\ln 1 - \ln 1)\Big]$$

$$= \frac{1}{2}\left[\ln\frac{3}{2} - \ln\frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\ln\frac{3}{2} - \ln\frac{1}{2}\right]$$
$$= \frac{1}{2}\ln 3$$

c) $|t| = \sqrt{1+9+4} = \sqrt{14}$

$|u| = \sqrt{16+1+4} = \sqrt{21}$

$|v| = \sqrt{25+4+49} = \sqrt{78}$

For the three sides to form a triangle,

$|t| + |u| \geq |v|$

but $|t| + |u| = \sqrt{14} + \sqrt{21}$

$= 8.324$

$= \sqrt{69.292\ldots}$

$< \sqrt{78}$

$\therefore$ so they cannot form a triangle.

d) $$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\,dx$$

| $f(x)$ | $f(\pi - x)$ |
| --- | --- |
| $x$ | $\pi - x$ |
| $\sin x$ | $\sin(\pi - x) = \sin x$ |
| $\cos^2 x$ | $\cos^2(\pi - x) = \cos^2 x$ |

$$= \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$$

$$\therefore 2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$\therefore \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = -\frac{\pi}{2}\int_0^{\pi} \frac{-\sin x}{1+\cos^2 x}$$

$$= -\frac{\pi}{2}\left[\tan^{-1}\cos x\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right]$$

$$= \frac{\pi^2}{4}$$

e) i Jimin didn't claim that $f(n)$ produced all prime numbers, just that all numbers produced are prime.

ii $f(41) = 41^2 + 41 + 41$, which clearly has 41 as a factor.

Question 15

a) i $I_n = \int x(\ln x)^n \, dx$

$u = (\ln x)^n \qquad v = \frac{1}{2}x^2$

$u' = n \times \frac{1}{x} \times (\ln x)^{n-1} \qquad v' = x$

$= uv - \int u'v \, dx$

$= \frac{1}{2}x^2(\ln x)^n - \int \frac{1}{2}x^2 \times \frac{1}{x} \times n \times (\ln x)^{n-1} \, dx$

$= \frac{1}{2}x^2(\ln x)^n - \frac{1}{2}n\int x(\ln x)^{n-1} dx$

$= \frac{1}{2}x^2(\ln x)^n - \frac{1}{2}n I_{n-1}$

ii $I_0 = \int_1^3 x(\ln x)^0 \, dx$

$= \left[\frac{x^2}{2}\right]_1^3$

$= \frac{9}{2} - \frac{1}{2}$

$= 4$

$I_1 = \int_1^3 x \ln x \, dx$

$= \left[\frac{1}{2}x^2 \ln x\right]_1^3 - \frac{1}{2}I_0$

$= \frac{1}{2}\left[9\ln 3 - \ln 1\right] - \frac{1}{2} \times 4$

$= \frac{9}{2}\ln 3 - 2$

$I_2 = \int_1^3 x(\ln x)^2 \, dx$

$= \left[\frac{1}{2}x^2(\ln x)^2\right]_1^3 - \frac{1}{2} \times 2 \times I_1$

$= \frac{1}{2}\left[9(\ln 3)^2\right] - \left[\frac{9}{2}\ln 3 - 2\right]$

$= \frac{9}{2}(\ln 3)^2 - \frac{9}{2}\ln 3 + 2$

b) i $r_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \lambda_1 \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} \qquad r_2 = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$

$x_1 = 1 + 2\lambda_1$ (1)

$y_1 = 2\lambda_1$ (2)

$z_1 = 2 - 3\lambda_1$ (3)

$x_2 = 4 + \lambda_2$ (4)

$y_2 = -2 + 2\lambda_2$ (5)

$y_3 = 9 - 2\lambda_2$ (6)

(1) = (4)

$1 + 2\lambda_1 = 4 + \lambda_2$ (1A)

(2) = (5)

$2\lambda_1 = -2 + 2\lambda_2$ (2A)

sub (2A) into (1A)

$1 + (-2 + 2\lambda_2) = 4 + \lambda_2$

$-1 + 2\lambda_2 = 4 + \lambda_2$

$\lambda_2 = 5$

when $\lambda_2 = 5$, $\quad 2\lambda_1 = -2 + 2(5)$
$$= 8$$
$$\therefore \lambda_1 = 4$$

when $\lambda_1 = 4$, $\lambda_2 = 5$

$x_1 = 9 \qquad x_2 = 9$

$y_1 = 8 \qquad y_2 = 8$

$z_1 = -10 \qquad z_2 = -1$, $\quad \therefore$ they do not intersect

ii

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(4 + \lambda_2 - 1 - 2\lambda_1)^2 + (-2 + 2\lambda_2 - 2\lambda_1)^2 + (9 - 2\lambda_2 - 2 + 3\lambda_1)^2}$$

$$= \sqrt{(3 + \lambda_2 - 2\lambda_1)^2 + (-2 + 2\lambda_2 - 2\lambda_1)^2 + (7 - 2\lambda_2 + 3\lambda_1)^2}$$

$$= \sqrt{\begin{aligned} &3^2 + \lambda_2^2 + 4\lambda_1^2 + 2(3\lambda_2 - 6\lambda_1 - 2\lambda_1\lambda_2) \\ &+ 4 + 4\lambda_2^2 + 4\lambda_1^2 + 2(-4\lambda_2 + 4\lambda_1 - 4\lambda_1\lambda_2) \\ &+ 49 + 4\lambda_2^2 + 9\lambda_1^2 + 2(-14\lambda_2 + 21\lambda_1 - 6\lambda_1\lambda_2) \end{aligned}}$$

$$= \sqrt{9\lambda_2^2 + 17\lambda_1^2 + 62 + \lambda_1(-12 + 8 + 42) + \lambda_2(6 - 8 - 28) + \lambda_1\lambda_2(-4 - 8 - 12)}$$

$$= \sqrt{9\lambda_2^2 + 17\lambda_1^2 + 62 + 38\lambda_1 - 30\lambda_2 - 24\lambda_1\lambda_2}$$

$$= \sqrt{9\lambda_2^2 + 16\lambda_1^2 + 40\lambda_1 - 30\lambda_2 - 24\lambda_1\lambda_2 + 25 + (\lambda_1^2 - 2\lambda + 1) + 36}$$

$$= \sqrt{9\lambda^2 + 16\lambda_1^2 + 25 - 2(12\lambda_1\lambda_2 + 15\lambda_2 - 20\lambda_1) + (\lambda_1 - 1)^2 + 36}$$

$$= \sqrt{(3\lambda_2 - 4\lambda_1 - 5)^2 * + (\lambda_1 - 1)^2 + 36}$$

$*$ by the result given.

iii Minimum distance when $(\lambda_1 - 1) = 0$ and $(3\lambda_2 - 4\lambda_1 - 5) = 0$

$$\therefore d = \sqrt{0^2 + 0^2 + 36}$$
$$= 6$$

iv $\lambda_1 - 1 = 0$
$$\therefore \lambda_1 = 1$$

$$3\lambda_2 - 4 - 5 = 0$$
$$3\lambda_2 = 9$$
$$\lambda_2 = 3$$

when $\lambda_1 = 1$, $\begin{pmatrix} 1+2 \\ 0+2 \\ 2-3 \end{pmatrix} \Rightarrow (3, 2, -1)$ $\qquad$ when $\lambda_2 = 3$ $\begin{pmatrix} 4+3 \\ -2+6 \\ 9-6 \end{pmatrix} \Rightarrow (7, 4, 3)$

Question 16

a) The proof will be by contrapositive

if $r$ is irrational $\Rightarrow \sqrt{r}$ is irrational

CONTRAPOSITIVE: if $\sqrt{r}$ is rational, then $r$ is rational

if $\sqrt{r}$ is rational, then let $\sqrt{r} = \frac{a}{b}$ $\quad a, b \in \mathbb{Z}$

$\therefore r = \frac{a^2}{b^2}$

$\therefore r$ is also rational

$\therefore$ if $\sqrt{r}$ is rational, then $r$ is rational

$\therefore$ if $r$ is irrational, then $\sqrt{r}$ is also irrational

b) let $i = e^{i\frac{\pi}{2}}$

$\therefore i^i = \left(e^{i\frac{\pi}{2}}\right)^i$

$= e^{i^2\frac{\pi}{2}}$

$= e^{-\frac{\pi}{2}}$

$\doteqdot 0.207$

which is real.

c) $t = \tan\frac{x}{2}$

i $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\therefore \frac{dx}{dt} = \frac{2}{\sec^2\frac{x}{2}}$

$= \frac{2}{1+\tan^2\frac{x}{2}}$

$= \frac{2}{1+t^2}$

ii $\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}} \begin{matrix}\times\cos^2\frac{x}{2} \\ \times\cos^2\frac{x}{2}\end{matrix}$

$= 2\tan\frac{x}{2} \times \cos^2\frac{x}{2}$

$= 2\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} \times \cos^2\frac{x}{2}$

$= 2\sin\frac{x}{2}\cos\frac{x}{2}$

$= \sin\left(\frac{x}{2}+\frac{x}{2}\right)$

$= \sin x$

$\int \frac{f'(x)}{a^2+[f(x)]^2}dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a}$

iii $\int_0^{\frac{\pi}{2}} \frac{1}{1+k\sin x}dx$

let $t = \tan\frac{x}{2}$

$\therefore \frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2dt}{1+t^2}$

$k\sin x = \frac{2kt}{1+t^2}$

$t = \tan\frac{x}{2}$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$\sin x = \frac{2t}{1+t^2}$

$= \int_0^1 \frac{1}{1+\left(\frac{2kt}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\,dt$

$$= 2\int_0^1 \frac{1}{\left(\frac{1+t^2+2kt}{1+t^2}\right)} \times \frac{1}{1+t^2}\, dt$$

$k\sin x = \frac{2kt}{1+t^2}$

when $x=0,\ t=0$

$x=\frac{\pi}{2},\ t=1$

$\sin x = \frac{\ }{1+t^2}$

$$= 2\int_0^1 \frac{1}{t^2+2kt+1}\, dt$$

$$= 2\int_0^1 \frac{1}{t^2+2kt+k^2+1-k^2}\, dt$$

$$= 2\int_0^1 \frac{1}{(t+k)^2+(1-k^2)}\, dt$$

$$= 2\left[\frac{1}{\sqrt{1-k^2}}\tan^{-1}\left(\frac{t+k}{\sqrt{1-k^2}}\right)\right]_0^1$$

$$= \frac{2}{\sqrt{1-k^2}}\left[\tan^{-1}\frac{1+k}{\sqrt{(1-k^2)}} - \tan^{-1}\frac{k}{\sqrt{1-k^2}}\right]$$

$$= \frac{2}{\sqrt{1-k^2}}\tan^{-1}\frac{\sqrt{1-k}}{\sqrt{1+k}}$$

$$= \frac{2}{\sqrt{1-k^2}}\tan^{-1}\sqrt{\frac{1-k}{1+k}}$$

$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$

$(A-B) = \tan^{-1}\left(\frac{\tan A - \tan B}{1+\tan A\tan B}\right)$

let $x = \tan^{-1}\frac{1+k}{\sqrt{1-k^2}} - \tan^{-1}\frac{k}{\sqrt{1-k^2}}$

$$\tan x = \tan\left(\tan^{-1}\frac{1+k}{\sqrt{1-k^2}} - \tan^{-1}\frac{k}{\sqrt{1-k^2}}\right)$$

$$= \frac{\frac{1+k}{\sqrt{1-k^2}} - \frac{k}{\sqrt{1-k^2}}}{1+\frac{1+k}{\sqrt{1-k^2}}\times\frac{k}{\sqrt{1-k^2}}}$$

$$= \frac{\frac{1}{\sqrt{1-k^2}}}{1+\frac{k+k^2}{1-k^2}}$$

$$= \frac{\frac{1}{\sqrt{1-k^2}}}{\frac{1-k^2+k+k^2}{1-k^2}} \quad \times(1-k^2)\ \times(1-k^2)$$

$$= \frac{\sqrt{1-k^2}}{1+k}$$

$$= \frac{\sqrt{1-k}\sqrt{1+k}}{1+k}$$

$$= \frac{\sqrt{1-k}}{\sqrt{1+k}}$$

$$\therefore x = \tan^{-1}\frac{\sqrt{1-k}}{\sqrt{1+k}}$$

iv $\quad I_{n+1} + 2I_n = \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1}x}{2+\sin x}\, dx + 2\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{2+\sin x}$

$\sin^n x \sin x$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^{n+1}x + 2\sin^n x}{2+\sin x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^n x(\sin x+2)}{2+\sin x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^n x\, dx$$

v $\quad$ let $n=1$

$$I_2 + 2I_1 = \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$\therefore I_2 = \int_0^{\frac{\pi}{2}} \sin x - 2I_1$$

$I = 1 - 2\left[\frac{\pi}{\ } \ldots \left[\frac{\pi}{\ }\right]\right.$

$$\therefore I_2 = \int_0^{\frac{\pi}{2}} \sin x \; - 2I_1$$
$$= \left[-\cos x\right]_0^{\frac{\pi}{2}} - 2I_1$$
$$= -\left[0 - 1\right] - 2I_1$$
$$= 1 - 2I_1$$

also when $n=0$

$$I_1 + 2I_0 = \int_0^{\frac{\pi}{2}} 1 \, dx$$
$$I_1 = \left[x\right]_0^{\frac{\pi}{2}} - 2I_0$$
$$= \frac{\pi}{2} - 2I_0$$

$$I_2 = 1 - 2\left[\frac{\pi}{2} - 2\left[\frac{\pi}{3\sqrt{3}}\right]\right]$$
$$= 1 - \pi + \frac{4\pi}{3\sqrt{3}}$$
$$= 1 - \frac{9\pi}{9} + \frac{4\sqrt{3}\pi}{9}$$
$$= \left(\frac{4\sqrt{3}-9}{9}\right)\pi + 1$$

$$I_0 = \int_0^{\frac{\pi}{2}} \frac{(\sin x)^0}{2+\sin x} dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x} dx$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{2}\sin x} dx$$
$$= \frac{1}{2} \times \frac{2}{\sqrt{1-\frac{1}{4}}} \tan^{-1}\sqrt{\frac{\frac{1}{2}}{1\frac{1}{2}}}$$
$$= \frac{1}{\sqrt{\frac{3}{4}}} \tan^{-1}\sqrt{\frac{1}{3}}$$
$$= \frac{2}{\sqrt{3}} \times \frac{\pi}{6}$$
$$= \frac{\pi}{3\sqrt{3}}$$